

Intergenerational Transfers, Lifetime Welfare and Resource Preservation: Mathematical Appendix

Simone Valente
Center of Economic Research, ETH Zurich.

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Address for correspondence:
Dr. Simone Valente
CER-ETH Zurich
Zurichbergstrasse 18, ZUE-F15
CH-8032 Zurich (Switzerland)
email: svalente@ethz.ch

1 The basic model (section 2)

The consumer problem. From (3), (4), and (13), the second-period individual constraint (9) can be rewritten as $e_{t+1} = q_{t+1} (1 + \varepsilon) a_t$, which can be substituted in (10) to obtain

$$c_t = w_t - \frac{q_t (1 - d_t) e_{t+1}}{q_{t+1} (1 + \varepsilon)}. \quad (\text{A1})$$

The individual problem consists of choosing c_t and e_{t+1} in order to maximize lifetime utility subject to (A1): first order conditions for an interior solution imply (14). Substituting equilibrium prices (7)-(8) and condition (14) in individual budget constraints (9)-(10), equilibrium consumption levels are

$$c_t = \frac{w_t}{1 + \beta} = \frac{1}{1 + \beta} (1 - \alpha) y_t, \quad (\text{A2})$$

$$e_{t+1} = \frac{1 + n}{1 + \beta} (\alpha + \beta) y_{t+1}. \quad (\text{A3})$$

Derivation of (15). Substituting $e_{t+1} = q_{t+1} (1 + \varepsilon) a_t$ in (A3) gives

$$a_t = \frac{(1 + n) (\alpha + \beta)}{q_{t+1} (1 + \beta) (1 + \varepsilon)} y_{t+1}. \quad (\text{A4})$$

From (7) and (13), $q_{t+1} = \alpha m_{t+1}^\alpha x_{t+1}^{\alpha-1} (1 - \tau_{t+1})$ can be substituted in (A4) to obtain

$$a_t = \frac{(1 + n) (\alpha + \beta)}{\alpha (1 + \beta) (1 + \varepsilon) (1 - \tau_{t+1})} x_{t+1}. \quad (\text{A5})$$

Now consider the system

$$\frac{q_{t+1}}{q_t} = \frac{e_{t+1} (1 - d_t)}{\beta c_t (1 + \varepsilon)}, \quad (\text{A6})$$

$$\frac{q_{t+1}}{q_t} = \frac{p_{t+1}}{p_t} \left(\frac{1 - \tau_{t+1}}{1 - \tau_t} \right), \quad (\text{A7})$$

where (A6) is the optimality condition (14), and (A7) is implied by no-arbitrage condition (13). Substituting (A2)-(A3) in (A6), and (7) in (A7) respectively gives

$$\frac{q_{t+1}}{q_t} = \frac{(1 + n) (\alpha + \beta) (1 - d_t)}{\beta (1 - \alpha) (1 + \varepsilon)} \left(\frac{y_{t+1}}{y_t} \right), \quad (\text{A8})$$

$$\frac{q_{t+1}}{q_t} = \frac{x_t}{x_{t+1}} \left(\frac{1 - \tau_{t+1}}{1 - \tau_t} \right) \frac{y_{t+1}}{y_t}, \quad (\text{A9})$$

implying

$$\frac{x_{t+1}}{x_t} = \frac{\beta (1 - \alpha) (1 + \varepsilon) (1 - \tau_{t+1})}{(1 + n) (\alpha + \beta) (1 - d_t) (1 - \tau_t)}. \quad (\text{A10})$$

Substituting (A10) in (A5) gives eq.(15) in the text.

Proof of Proposition 1. Under laissez-faire $z_{t+1} = z_t = \tilde{z}$, which implies that U_t is proportional to y_t (see equation (A14) derived below). Hence, satisfying the sustainability condition (1) in the laissez-faire economy requires $\theta^y \geq 1$. Setting $z_{t+1} = z_t = \tilde{z}$ in (18) it follows that $\theta^y \geq 1$ if and only if (22) is satisfied. Substituting (15) and $\gamma = \beta^{-1} - 1$ in (22) yields (23).

Deriving tax-subsidy rates in (25). Setting $z_t = z'$, $\tau_t = \tau'$ and $d_t = d'$ in equations (24) and (11) gives

$$z' = \tilde{z} (1 - \tau') (1 - d'), \quad (\text{A11})$$

$$\tau' z' = (1 - \tau') d', \quad (\text{A12})$$

respectively. Substituting (A12) in (A11) gives $\tau' = \frac{d'}{\bar{z}(1-d')}$, which can be substituted back in (A11) to obtain $d' = \frac{\bar{z}-z'}{1+\bar{z}}$, which is the subsidy rate level in (25). The tax rate level in (25) then follows from $\tau' = \frac{d'}{\bar{z}(1-d')}$ as obtained above.

Proof of Lemma 2. It follows from (A2)-(A3) that

$$U_t = \log \left(\frac{1-\alpha}{1+\beta} \right) + \beta \log(1+n) \frac{\alpha+\beta}{1+\beta} + \log y_t + \beta \log y_{t+1}. \quad (\text{A13})$$

From (18), $\beta \log y_{t+1} = \beta \log y_t + \alpha\beta \log \frac{z_{t+1}(1+\rho)}{z_t(1+z_{t+1})}$, and (A13) can be rewritten as

$$U_t = \log \left\{ \left(\frac{1-\alpha}{1+\beta} \right) \left[\frac{(1+n)(\alpha+\beta)}{1+\beta} \right]^\beta \left[\frac{z_{t+1}(1+\rho)}{z_t(1+z_{t+1})} \right]^{\alpha\beta} y_t^{1+\beta} \right\}. \quad (\text{A14})$$

If the policymaker sets $z_t = \rho$ in each period, (18) implies y_t be constant over time and (A14) implies U_t be constant over time. More generally, from (A14), any path with constant utility requires

$$\frac{z_{t+2}}{1+z_{t+2}} = \frac{z_{t+1}}{(1+\rho)} \left[\frac{z_t(1+z_{t+1})}{z_{t+1}(1+\rho)} \right]^{\frac{1}{\beta}} \text{ for each } t \in [0, \infty). \quad (\text{A15})$$

2 Resource-saving transfers (section 3)

Deriving expression (28). Given the initial endowment $R_0 \equiv r_0 N_0$, solving (16) and (17) backward yields

$$x_t = r_0 \left(\frac{1+\varepsilon}{1+n} \right)^t \cdot \frac{z_t}{1+z_t} \prod_{j=0}^{t-1} \frac{1}{1+z_j}. \quad (\text{A16})$$

Substituting (A16) in $y_t = m_t^\alpha x_t^\alpha$ gives

$$y_t = \left(\frac{z_t}{1+z_t} \phi_t \right)^\alpha, \quad (\text{A17})$$

where

$$\phi_t \equiv \frac{r_0 m_0 (1+\rho)^t}{\prod_{j=0}^{t-1} (1+z_j)} \quad (\text{A18})$$

is a function of H_t and is therefore taken as given by the agent born in period t . Expression (A18) implies that $\phi_{t+1} = \phi_t (1+\rho) (1+z_t)^{-1}$, and thus

$$y_{t+1} = \left[\frac{(1+\rho) z_{t+1}}{(1+z_t)(1+z_{t+1})} \phi_t \right]^\alpha. \quad (\text{A19})$$

Substituting (A17) and (A19) in (A13) yields

$$U_t = \log \left\{ \left[\left(\frac{1-\alpha}{1+\beta} \right) \left(\frac{z_t}{1+z_t} \phi_t \right) \right]^\alpha \left[\frac{(1+n)(\alpha+\beta)}{1+\beta} \right]^\beta \left[\frac{(1+\rho) z_{t+1}}{(1+z_t)(1+z_{t+1})} \phi_t \right]^{\alpha\beta} \right\}.$$

Setting $\Omega_t \equiv \log \left\{ \left(\frac{1-\alpha}{1+\beta} \right) \phi_t^\alpha \left[\frac{(1+n)(\alpha+\beta)}{1+\beta} \phi_t^\alpha \right]^\beta \right\}$ yields expression (28) in the text.

Deriving expressions (29) and (30). It follows from (28) that

$$V(0,0) = \Omega + \log \left\{ \left(\frac{\tilde{z}}{1+\tilde{z}} \right)^\alpha \left(\frac{1}{1+\tilde{z}} \right)^{\alpha\beta} \left[\frac{(1+\rho)\tilde{z}}{1+\tilde{z}} \right]^{\alpha\beta} \right\}, \quad (\text{A20})$$

$$V(0,1) = \Omega + \log \left\{ \left(\frac{\tilde{z}}{1+\tilde{z}} \right)^\alpha \left(\frac{1}{1+\tilde{z}} \right)^{\alpha\beta} \left[\frac{(1+\rho)z'}{1+z'} \right]^{\alpha\beta} \right\}, \quad (\text{A21})$$

$$V(1,0) = \Omega + \log \left\{ \left(\frac{z'}{1+z'} \right)^\alpha \left(\frac{1}{1+z'} \right)^{\alpha\beta} \left[\frac{(1+\rho)\tilde{z}}{1+\tilde{z}} \right]^{\alpha\beta} \right\}, \quad (\text{A22})$$

$$V(1,1) = \Omega + \log \left\{ \left(\frac{z'}{1+z'} \right)^\alpha \left(\frac{1}{1+z'} \right)^{\alpha\beta} \left[\frac{(1+\rho)z'}{1+z'} \right]^{\alpha\beta} \right\}. \quad (\text{A23})$$

Expressions (29) and (30) in the text are proved as follows: $\tilde{z} > z'$ implies

$$\frac{z'}{\tilde{z}} \left(\frac{1+\tilde{z}}{1+z'} \right) < 1. \quad (\text{A24})$$

Hence, from (A20)-(A21) we have $V(0,0) > V(0,1)$, because $\left[\frac{\tilde{z}(1+z')}{z'(1+\tilde{z})} \right]^\alpha > 1$; from (A22)-

(A23) we have $V(1,0) > V(1,1)$, because $\left[\frac{\tilde{z}(1+z')}{z'(1+\tilde{z})} \right]^{\alpha\beta} > 1$.

Proof of Proposition 3. From (A20) and (A23), $V(0,0) < V(1,1)$ if and only if

$$\left(\frac{\tilde{z}}{1+\tilde{z}} \right)^{\alpha(1+\beta)} \left(\frac{1}{1+\tilde{z}} \right)^{\alpha\beta} < \left(\frac{z'}{1+z'} \right)^{\alpha(1+\beta)} \left(\frac{1}{1+z'} \right)^{\alpha\beta}.$$

Substituting $1+\tilde{z} = \frac{\beta+\alpha}{\beta(1-\alpha)}$, this inequality reduces to (32). It follows from (29) and (30) that, if (32) is satisfied, the only possible payoff ranking is (31).

Deriving expression (33). From (A20) and (A23), the gap $\Phi = V(0,0) - V(1,1)$ equals

$$\Phi = \log \left\{ \left(\frac{\tilde{z}}{z'} \right)^{\alpha+\alpha\beta} \left(\frac{1+z'}{1+\tilde{z}} \right)^{\alpha+2\alpha\beta} \right\}.$$

Substituting $z' = \mu\tilde{z}$ and eq.(15) in the above expression yields equation (33) in the text.

The Rawls-optimal path. Along a Rawls-optimal path $U_{t+1} = U_t = U^*$ in any $t \geq 0$, where U^* is the maximum utility level that can be sustained indefinitely. Utility per capita is constant if $y_t = y^*$ at any $t \geq 0$, implying $m_t^\alpha x_t^\alpha = y^*$. Defining $\hat{x}_t \equiv m_t x_t$, the Rawls-optimal path requires $\hat{x}_t = \hat{x}^* = (y^*)^{\frac{1}{\alpha}}$ constant. Multiplying by m_t the resource constraint $r_{t+1}(1+n) = (1+\varepsilon)(r_t - x_t)$, and defining $\tilde{r}_t \equiv r_t m_t$, we obtain

$$\tilde{r}_{t+1} = (1+\rho)(\tilde{r}_t - \hat{x}^*). \quad (\text{A25})$$

Equation (A25) has a unique steady state

$$\tilde{r}^* = \frac{1+\rho}{\rho} \hat{x}^*, \quad (\text{A26})$$

which is unstable since $\rho > 0$. Setting $\tilde{x}^* > \tilde{r}_0 \rho (1+\rho)^{-1}$ we would have $\tilde{r}_t \rightarrow -\infty$, which is unfeasible; setting $\tilde{x}^* < \tilde{r}_0 \rho (1+\rho)^{-1}$ we would have $\tilde{r}_t \rightarrow +\infty$, which is feasible but technologically inefficient (there would be waste of productive resources). Hence, the Rawls-optimal plan features $\tilde{x}^* = \tilde{r}_0 \rho (1+\rho)^{-1}$, which implies $x_0^* = r_0 \rho (1+\rho)^{-1}$. Since x_t^* must decline geometrically at rate δ to ensure constancy of \tilde{x}^* , natural capital and the resource stock evolve according to expression (34) in the text. Hence, output per capita

is $y^* = \left[m_0 r_0 \rho (1 + \rho)^{-1} \right]^\alpha$ at each $t \geq 0$. Given y^* , lifetime utility is maximized by the consumption bundles

$$(c^*, e^*) = \arg \max \left\{ \log c^* + \beta \log e^* \text{ sub } y^* = c^* + e^* (1 + n)^{-1} \right\}.$$

Substituting the resulting first-order condition $e^* = \beta c^* (1 + n)$ in the aggregate budget constraint gives conditions (35), which are the same as in Mourmouras (1993).

Also the first-best policy is analogous to Mourmouras (1993). At time zero, the resource stock is expropriated and a stock J_0 of fiat currency is introduced in the economy as a lump-sum transfer to the initial old. Denoting the quantity of money per young individual as $j_t = J_0/N_t$, the first father receives (and consumes)

$$\frac{j_t (1 + n)}{p_t^y} = e_0,$$

where p_t^y is the price index of the economy. All successive generations face individual constraints

$$c_t = w_t + s_t - (j_t/p_t^y), \quad (\text{A27})$$

$$e_{t+1} = (j_t/p_{t+1}^y) \quad (\text{A28})$$

where s_t is the lump-sum subsidy through which the government transfers all rents from natural capital to the currently young. Hence, consumers maximize U_t subject to (A27)-(A28), obtaining first-order conditions

$$e_{t+1} = \beta c_t (p_t^y/p_{t+1}^y). \quad (\text{A29})$$

Since the government is implementing the depletion paths (34), transfers to the young equal $s_t = \alpha y^*$ in each period: substituting this amount in the budget constraints (A27)-(A28) together with the first-order condition (A29), we obtain

$$\begin{aligned} c_t &= (1 + \beta)^{-1} y^* = c^*, \\ e_t &= \beta (1 + n) (1 + \beta)^{-1} y^* = e^*, \end{aligned}$$

which is indeed the Rawls-optimal allocation (35).

3 Man-made capital (section 4.1)

Derivation of (43) and (44). Output is now given by $Y = X^{\alpha_1} N^{\alpha_2} K^{\alpha_3}$ and profit-maximizing conditions read

$$p_t = \alpha_1 (y_t/x_t), \quad w_t = \alpha_2 y_t, \quad i_t = \alpha_3 (y_t/k_t). \quad (\text{A30})$$

The individual budget constraints read

$$c_t = w_t - q_t a_t (1 - d) - k_{t+1}, \quad (\text{A31})$$

$$e_{t+1} = q_{t+1} a_{t+1} + p_{t+1} (1 - \tau) x_{t+1} + i_{t+1}^k k_{t+1}. \quad (\text{A32})$$

From (A31)-(A32), the present-value budget constraint of each consumer reads

$$c_t + \frac{e_{t+1}}{i_{t+1}^k} = w_t - q_t a_t (1 - d) + \frac{1}{i_{t+1}^k} [q_{t+1} a_{t+1} + p_{t+1} (1 - \tau) x_{t+1}]. \quad (\text{A33})$$

Substituting $p_{t+1} (1 - \tau) = q_{t+1}$ from (13) and $a_{t+1} + x_{t+1} = r_{t+1} = (1 + \varepsilon) a_t$ from (4),

$$c_t + \frac{e_{t+1}}{i_{t+1}^k} = w_t + \frac{a_t}{i_{t+1}^k} [q_{t+1} (1 + \varepsilon) - q_t (1 - d)]. \quad (\text{A34})$$

Maximization of lifetime income in (A34) yields the Hotelling rule

$$\frac{q_{t+1}}{q_t} = i_{t+1}^k \left(\frac{1-d}{1+\varepsilon} \right), \quad (\text{A35})$$

whereas maximization of lifetime utility subject to (A34) gives the Euler condition

$$e_{t+1} = \beta c_t i_{t+1}^k. \quad (\text{A36})$$

Expressions (A35)-(A36) imply conditions (44) in the text. Substituting (A35)-(A36) back in (A34) yields

$$c_t = \frac{w_t}{1+\beta} = \frac{\alpha_2}{1+\beta} y_t, \quad (\text{A37})$$

which is condition (43) in the text.

Derivation of (45). Substituting (A37) in (A31),

$$q_t a_t (1-d) + k_{t+1} = w_t - c_t = \frac{\alpha_2 \beta}{1+\beta} y_t. \quad (\text{A38})$$

Using $q_t = p_t (1-\tau)$ and $p_t = \alpha_1 (y_t/x_t)$, net expenditure in resource assets equals

$$q_t a_t (1-d) = \frac{\alpha_1 (1-\tau) (1-d)}{z_t} y_t. \quad (\text{A39})$$

Plugging (A39) in (A38), we get

$$\frac{k_{t+1}}{y_t} = \frac{\alpha_2 \beta}{1+\beta} - \frac{\alpha_1 (1-\tau) (1-d)}{z_t}. \quad (\text{A40})$$

On the other hand, substituting $i_{t+1}^k = \alpha_3 (y_{t+1}/k_{t+1})$ and $p_t = \alpha_1 (y_t/x_t)$ in (A35), we obtain

$$\frac{k_{t+1}}{y_t} = \alpha_3 \left(\frac{x_{t+1}}{x_t} \right) \left(\frac{1-d}{1+\varepsilon} \right). \quad (\text{A41})$$

Plugging (A41) in (A40), we get

$$\frac{\alpha_2 \beta}{1+\beta} - \frac{\alpha_1 (1-\tau) (1-d)}{z_t} = \alpha_3 \left(\frac{x_{t+1}}{x_t} \right) \left(\frac{1-d}{1+\varepsilon} \right). \quad (\text{A42})$$

A shortcut here is to note that z_t must be constant since $\frac{x_{t+1}}{x_t}$ only depends on z_t : substituting $\frac{x_{t+1}}{x_t} = \frac{1+\varepsilon}{1+z}$ in (A42) indeed yields expression (45) in the text.

The fact that z_t is constant can also be verified from the government budget constraint (11) - which is balanced with constant tax-subsidy rates only if z_t is constant - or even proved as follows: rewrite (A42) as

$$\alpha_3 \left(\frac{x_{t+1}}{x_t} \right) \left(\frac{1-d}{1+\varepsilon} \right) = \frac{\alpha_2 \beta}{1+\beta} - \alpha_1 (1-\tau) (1-d) \left(\frac{r_t}{x_t} - 1 \right), \quad (\text{A43})$$

and consider the resource constraint: from $r_{t+1} = (1+\varepsilon)(r_t - x_t)$ and (A43) above, the dynamics of resource exploitation are reduced to the two-by-two system of difference equations

$$\begin{aligned} r_{t+1} &= a_{11} r_t + a_{12} x_t, \\ x_{t+1} &= a_{21} r_t + a_{22} x_t, \end{aligned}$$

where

$$\begin{aligned} a_{11} &= a_{12} = 1 + \varepsilon > 0, \\ a_{21} &= -(\alpha_1/\alpha_3) (1-\tau) (1+\varepsilon) < 0, \\ a_{22} &= \frac{\alpha_2 \beta + \alpha_1 (1+\beta) (1-d) (1-\tau)}{\alpha_3 (1+\beta) (1-d) (1+\varepsilon)^{-1}} > 0. \end{aligned}$$

Denoting by $\zeta_{1,2}$ the eigenvalues of the system, the characteristic equation is

$$\zeta^2 - (a_{11} + a_{22})\zeta + (a_{11}a_{22} - a_{12}a_{21}) = 0, \quad (\text{A44})$$

displaying roots

$$\zeta_{1,2} = (1/2) \left[a_{11} + a_{22} \pm \sqrt{(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21})} \right].$$

Denoting by ζ_* the stable root, the resource stock evolves according to $r_{t+1} = \zeta_* r_t$ with r_0 given. Substituting this solution in $r_{t+1} = (1 + \varepsilon)(r_t - x_t)$, and recalling that $z_t \equiv x_t/a_t$ we obtain $1 + z_t = (1 + \varepsilon)\zeta_*^{-1}$, which is constant (Q.E.D.). Note that multiplying both sides of (A44) by $\alpha_3(1 + \varepsilon)^{-1}$ we obtain (except for the presence of taxes and subsidies) the same characteristic equation in Mourmouras (1993: eq.(56), p.264), so that this procedure is equivalent.

Equilibrium dynamics (46). Given a constant depletion rate,

$$\theta_{t+1}^x = \theta_{t+1}^a = \theta_{t+1}^r = (1 + \varepsilon)(1 + z)^{-1},$$

whereas output and man-made capital evolve according to

$$\theta_{t+1}^k = \alpha_3 \left(\frac{1-d}{1+z} \right) x_t^{\alpha_1} k_t^{\alpha_3-1}, \quad (\text{A45})$$

$$\theta_{t+1}^y = (\theta_{t+1}^x)^{\alpha_1} (\theta_{t+1}^k)^{\alpha_3}. \quad (\text{A46})$$

where (A45) is obtained by substituting $y_t = x_t^{\alpha_1} k_t^{\alpha_3}$ in equation in (A41). From (A45) we have

$$\theta_{t+1}^k = \theta_t^y = (\theta_t^x)^{\alpha_1} (\theta_t^k)^{\alpha_3}, \quad (\text{A47})$$

which can be log-linearized as (defining $\bar{\theta}_t^i \equiv \log \theta_t^i$)

$$\bar{\theta}_{t+1}^k = \alpha_1 \bar{\theta}_t^x + \alpha_3 \bar{\theta}_t^k = \alpha_3 \bar{\theta}_t^k + \alpha_1 \log [(1 + \varepsilon) / (1 + z)]. \quad (\text{A48})$$

Since $\alpha_3 < 1$, $\bar{\theta}_t^k$ converges to the unique steady state

$$\bar{\theta}^k = \frac{\alpha_1}{1 - \alpha_3} \log [(1 + \varepsilon) / (1 + z)].$$

As a consequence,

$$\lim_{t \rightarrow \infty} \theta_t^k = [(1 + \varepsilon) / (1 + z)]^{\frac{\alpha_1}{1 - \alpha_3}}. \quad (\text{A49})$$

From (47), it derives that output and capital grow at the same rate in the long run, that is,

$$\lim_{t \rightarrow \infty} \theta_t^k = \lim_{t \rightarrow \infty} \theta_t^y.$$

Substituting (A46) yields expression (46) in the text. Also the interest rate is constant in the long run:

$$\lim_{t \rightarrow \infty} i_t = \lim_{t \rightarrow \infty} \alpha_3 (y_t/k_t) = (1 + z) \lim_{t \rightarrow \infty} \theta_t^y = (1 + z) [(1 + \varepsilon) / (1 + z)]^{\frac{\alpha_1}{1 - \alpha_3}}. \quad (\text{A50})$$

Critical condition (47). Using (A36) and (A37), lifetime utility of agents born in t equals

$$U_t = \log \left(\frac{\alpha_2}{1 + \beta} y_t \right) + \beta \log \left[\frac{y_t}{k_{t+1}} \left(\frac{\alpha_3 \alpha_2 \beta}{1 + \beta} \right) y_{t+1} \right]. \quad (\text{A51})$$

Setting $e_{t+1} = i_{t+1}\beta c_t = \frac{\alpha_2\alpha_3\beta}{1+\beta} \left(\frac{y_t}{k_{t+1}}\right) y_{t+1}$ in period t and substituting in the aggregate constraint $k_{t+1} = y_t - c_t - e_t$ yields $k_{t+1} = \bar{\varphi}_t y_t$, where

$$\bar{\varphi}_t \equiv 1 - \frac{\alpha_2}{1+\beta} - \frac{\alpha_2\alpha_3\beta}{1+\beta} \left(\frac{y_{t-1}}{k_t}\right)$$

is taken as given by agents born in period t . Hence, (A51) can be rewritten as

$$U_t = \log \left\{ \frac{\alpha_2}{1+\beta} \left[\frac{\alpha_3\alpha_2\beta}{\bar{\varphi}_t(1+\beta)} \right]^\beta \right\} + \log y_t + \beta \log y_{t+1}. \quad (\text{A52})$$

Using the definition of ϕ_t in (A18), output at subsequent dates can be written as¹

$$y_t = \left(\frac{z_t}{1+z_t}\right)^{\alpha_1} \phi_t^{\alpha_1} k_t^{\alpha_3}, \quad (\text{A53})$$

$$y_{t+1} = \left(\frac{z_t(1+\varepsilon)}{(1+z_t)(1+z_{t+1})}\right)^{\alpha_1} \phi_t^{\alpha_1} \bar{\varphi}_t^{\alpha_3} y_t^{\alpha_3}. \quad (\text{A54})$$

where we have used $k_{t+1}^{\alpha_3} = (\bar{\varphi}_t y_t)^{\alpha_3}$. Substituting (A53)-(A54) in (A52),

$$U_t = \Omega'_t + \log \left\{ \left(\frac{z_t}{1+z_t}\right)^{\alpha_1} \left[\frac{1+\varepsilon}{1+z_{t+1}} \left(\frac{z_t}{1+z_t}\right)^{(1+\alpha_3)} \right]^{\alpha_1\beta} \right\}, \quad (\text{A55})$$

where we have defined

$$\Omega'_t \equiv \log \left[\frac{\alpha_2\phi_t^{\alpha_1} k_t^{\alpha_3}}{1+\beta} \left(\frac{\alpha_3\alpha_2\beta\phi_t^{\alpha_1(1+\alpha_3)} \bar{\varphi}_t^{\alpha_3} k_t^{\alpha_3^2}}{\bar{\varphi}_t(1+\beta)} \right)^\beta \right].$$

Since Ω'_t is taken as given by agents born in t , the gap between utility under life-persistent transfers ($z_t = z_{t+1} = z'$) and under laissez-faire ($z_t = z_{t+1} = \tilde{z}$) is given by

$$\Phi = V(0,0) - V(1,1) = \log \left\{ \left[\frac{\tilde{z}}{z'} \left(\frac{1+z'}{1+\tilde{z}}\right) \right]^{\alpha_1+\alpha_1\beta(1+\alpha_3)} \left(\frac{1+z'}{1+\tilde{z}}\right)^{\alpha_1\beta} \right\}. \quad (\text{A56})$$

From (A56), setting $z' = \mu\tilde{z}$, the condition for $\Phi < 0$ is given by inequality (47) in the text. It should be stressed that the above expression for the welfare gap is valid at each point in time: although the model with man-made capital displays transitional dynamics in interest rates, we have derived (A56) by using intertemporal equilibrium relations, without imposing steady-state or long-run values for any endogenous variable. As a consequence, the critical condition for having a negative welfare gap is determined by the same inequality, $\Phi < 0$, in each period. The reason for this time-invariant condition is that z is constant over time, so that the intertemporal allocation of natural wealth for each agent along her lifecycle is determined by the same equilibrium condition independently of the time ($t = 1$ or $t \rightarrow \infty$) in which the agent happens to be alive.

4 Monopoly rents (section 4.2)

Derivation of (49)-(50)-(51). Given the production function

$$Y_t = X_t^{\nu_1} N^{\nu_2} \sum_{j=1}^g B_{(j),t}^{\nu_3},$$

¹With $\delta = n = 0$, ϕ defined in (A18) is here simplified by the fact that $\rho = \varepsilon$ and $m_0 = 1$.

final-good producers maximize profits with first order conditions

$$\begin{aligned} p_t X_t &= \nu_1 Y_t, \\ w_t L &= \nu_2 Y_t, \\ p_{(j),t}^b &= \nu_3 X_t^{\nu_1} L^{\nu_2} B_{(j),t}^{\nu_3-1}. \end{aligned} \quad (\text{A57})$$

Monopolistic producers of $B_{(j)}$ take the demand schedule (A57) as given and maximize profits

$$\bar{\pi}_{(j)} = B_{(j)} \left(p_{(j)}^b - 1 \right) = B_{(j)} \left(\nu_3 X_t^{\nu_1} L^{\nu_2} B_{(j)}^{\nu_3-1} - 1 \right), \quad (\text{A58})$$

obtaining first-order condition

$$B_t = B_{(j),t} = \left(\nu_3^2 X_t^{\nu_1} L^{\nu_2} \right)^{\frac{1}{1-\nu_3}}, \quad (\text{A59})$$

which is expression (49) in the text. Plugging (A59) in (A57) gives

$$p_t^b = p_{(j),t}^b = \frac{1}{\nu_3}, \quad (\text{A60})$$

and substituting (A60) in (A61) gives equilibrium profits

$$\bar{\pi}_t = \bar{\pi}_{(j),t} = \left(\frac{1-\nu_3}{\nu_3} \right) B_t. \quad (\text{A61})$$

Multiplying both sides of (A61) by g/N gives equation (50) in the text. Substituting (A59) in the aggregate production function gives (51).

Derivation of aggregate constraint (55). See derivation of (69) below.

Derivation of (56). Profit maximization implies $w_t = \nu_2 y_t$ and $p_t x_t = \nu_1 y_t$. Using these conditions and substituting (43) and $g b_t = \nu_3^2 y_t$ in (55) yields

$$e_{t+1} = \left[(1-\nu_3^2)(1+\beta) - \nu_2 \right] (1+\beta)^{-1} y_{t+1}. \quad (\text{A62})$$

From (A62) and (44), we obtain

$$\frac{x_{t+1}}{x_t} = \frac{\nu_2 \beta (1+\varepsilon) (1-d)^{-1}}{(1-\nu_3^2)(1+\beta) - \nu_2}, \quad (\text{A63})$$

which implies that z_t is constant over time (knife-edge equilibrium). Substituting $\theta^x = (1+\varepsilon)(1+z)^{-1}$ yields expression (56).

The gap function (58). Substituting (43) and (A62) in the utility function gives

$$U_t = \log \left\{ \frac{\nu_2}{1+\beta} \left[\frac{(1-\nu_3^2)(1+\beta) - \nu_2}{1+\beta} \right]^\beta \left[y_t^{1+\beta} \left(\frac{1+\varepsilon}{1+z} \right)^{\frac{\nu_1 \beta}{1-\nu_3}} \right] \right\} \quad (\text{A64})$$

where we have substituted y_{t+1} from (57). Substituting $x_t = r_t z (1+z)^{-1}$ and $b_t = (\nu_3^2 x_t^{\nu_1})^{\frac{1}{1-\nu_3}}$ in the production function yields

$$y_t = \frac{1}{\nu_3^2} g_t \left(\nu_3^2 r_t^{\nu_1} \right)^{\frac{1}{1-\nu_3}} \left(\frac{z}{1+z} \right)^{\frac{\nu_1}{1-\nu_3}}, \quad (\text{A65})$$

where r_t is taken as given by agents born in t . Substituting (A65) in (A64) we get

$$U_t = \Omega_t'' + \log \left[\left(\frac{z}{1+z} \right)^{\frac{\nu_1(1+\beta)}{1-\nu_3}} \left(\frac{1+\varepsilon}{1+z} \right)^{\frac{\nu_1 \beta}{1-\nu_3}} \right], \quad (\text{A66})$$

where

$$\Omega_t'' \equiv \log \left\{ \frac{\nu_2}{1+\beta} \left[\frac{(1-\nu_3^2)(1+\beta) - \nu_2}{1+\beta} \right]^\beta \left(\frac{1}{\nu_3^2} g \left(\nu_3^2 r_t^{\nu_1} \right)^{\frac{1}{1-\nu_3}} \right)^{1+\beta} \right\}$$

is historically determined. Setting $z = z'$ for $V(1,1)$ and $z = \tilde{z}$ for $V(0,0)$, the gap function is $\Phi = V(0,0) - V(1,1)$ can be expressed as in (58).

5 R&D activity (section 4.3)

Derivation of system (70)-(71). Substituting (64), (67) and $\pi_{t+1} = \nu_3^{-1}(1 - \nu_3)b_{t+1}$ in (68) we obtain

$$i_{t+1}^h = \frac{(g_{t+1} - g_t)(\pi_{t+1} + v_{t+1})}{h_t} = \xi_t [\nu_3^{-1}(1 - \nu_3)b_{t+1} + \xi_{t+1}^{-1}].$$

Substituting (63) and recalling that $y_t = \nu_3^{-2}g_t b_t$,

$$i_{t+1}^h = [1 + \psi\nu_3(1 - \nu_3)]\theta_{t+1}^b = [1 + \psi\nu_3(1 - \nu_3)](\theta_{t+1}^x)^{\frac{\nu_1}{1-\nu_3}}, \quad (\text{A67})$$

where we have used (61). Now rewrite the Hotelling rule (44) as

$$i_{t+1}^h = (1 + \varepsilon)\theta_{t+1}^y [(1 - d)\theta_{t+1}^x]^{-1}. \quad (\text{A68})$$

Plugging (A68) in (A67) and using $\theta_{t+1}^y = \theta_{t+1}^b \theta_{t+1}^g$, we get

$$\theta_{t+1}^x = \left(\frac{1 + \varepsilon}{1 - d} \right) \left[\frac{1 + \psi(h_t/y_t)}{1 + \psi\nu_3(1 - \nu_3)} \right], \quad (\text{A69})$$

where we have substituted $\theta_{t+1}^g = [1 + \psi(h_t/y_t)]$ from (62) and (63). The only endogenous variable in (A69) is the marginal propensity to invest in R&D, h_t/y_t , which can be obtained as follows. Rewrite (65) as

$$c_t/y_t = w_t - \nu_1(1 - d)(1 - \tau)z_t^{-1} - g_t v_t - (h_t/y_t),$$

and substitute (43) and (64) to obtain

$$h_t/y_t = \nu_2\beta(1 + \beta)^{-1} - \psi^{-1} - \nu_1(1 - d)(1 - \tau)z_t^{-1}. \quad (\text{A70})$$

which is equivalent to expression (71) in the text. Note that the marginal propensity to invest in existing firms, $f_t/y_t = 1/\psi$, must be less than unity in an interior equilibrium, so that positive R&D activity requires $\psi > 1$ as claimed in footnote 20. From (A69) and (A70), the growth rate of natural capital solely depends on z_t , which is therefore constant and obtained recursively: setting $\theta_{t+1}^x = (1 + \varepsilon)(1 + z)^{-1}$ in (A69) yields (70), to be combined with (A70) as shown in the main text.

The critical condition (74). From (43) and (44), lifetime utility equals

$$U_t = \log \left[c_t^{1+\beta} \beta^\beta (i_{t+1}^h)^\beta \right] = \log \left\{ \left[\nu_2(1 + \beta)^{-1} y_t \right]^{1+\beta} \beta^\beta (i_{t+1}^h)^\beta \right\}.$$

Substituting (A67) and $\theta^x = (1 + \varepsilon)(1 + z)^{-1}$,

$$U_t = \log \left\{ \beta^\beta [1 + \psi\nu_3(1 - \nu_3)]^\beta \left(\frac{\nu_2}{1 + \beta} \right)^{1+\beta} \left(\frac{1 + \varepsilon}{1 + z} \right)^{\frac{\nu_1\beta}{1-\nu_3}} y_t^{1+\beta} \right\}. \quad (\text{A71})$$

Substituting $x_t = r_t z(1 + z)^{-1}$ in (60), output reads

$$y_t = g_t x_t^{\nu_1} b_t^{\nu_3} = \nu_3^{\frac{2\nu_3}{1-\nu_3}} g_t x_t^{\frac{\nu_1}{1-\nu_3}} = g_t r_t^{\frac{\nu_1}{1-\nu_3}} \nu_3^{\frac{2\nu_3}{1-\nu_3}} \left(\frac{z}{1 + z} \right)^{\frac{\nu_1}{1-\nu_3}}. \quad (\text{A72})$$

Plugging (A72) in (A71) yields

$$U_t = \Omega_t''' + \log \left[\left(\frac{1 + \varepsilon}{1 + z} \right)^{\frac{\nu_1\beta}{1-\nu_3}} \left(\frac{z}{1 + z} \right)^{\frac{\nu_1}{1-\nu_3}(1+\beta)} \right], \quad (\text{A73})$$

where we have defined

$$\Omega_t''' \equiv \log \left\{ \beta^\beta [1 + \psi \nu_3 (1 - \nu_3)]^\beta \left(\nu_2 (1 + \beta)^{-1} g_t r_t^{\frac{\nu_1}{1-\nu_3}} \nu_3^{\frac{2\nu_3}{1-\nu_3}} \right)^{1+\beta} \right\},$$

which is taken as given by agents born in t . Hence, from (A73),

$$\Phi = V(0, 0) - V(1, 1) = \log \left[\left(\frac{\tilde{z}}{z'} \right)^{\frac{\nu_1}{1-\nu_3}(1+\beta)} \left(\frac{1+z'}{1+\tilde{z}} \right)^{(1+2\beta)\frac{\nu_1}{1-\nu_3}} \right]. \quad (\text{A74})$$

Setting $\mu = z'/\tilde{z}$, it follows from (A74) that $\Phi < 0$ when inequality (74) holds.

Derivation of aggregate constraint (69). Since the aggregate production function displays constant returns to scale, profit maximization of final output producers implies

$$y_t = p_t x_t + w_t + \nu_3^{-1} g_t b_t = p_t x_t + [c_t + q_t a_t (1 - d) + f_t + h_t] + \nu_3^{-1} g_t b_t,$$

where the term in square brackets follows from (65). Setting (66) at time t and substituting for $q_t a_t$ yields

$$y_t = c_t + e_t + h_t - g_t \pi_t + \nu_3^{-1} g_t b_t,$$

where we have simplified $p_t x_t \tau = q_t a_t d$ from (11). Substituting monopoly profits per capita $\pi_t = \nu_3^{-1} (1 - \nu_3) b_t$ yields (69). Setting $h_t = 0$ (no R&D investment) gives the aggregate constraint in the model with monopoly rents - expression (55).