2. Supply and Production Possibilities

2.1. Production functions

Basic concept: production function

$$X = F(K, L)$$

with
X = output
$F(.)$ = output technology
K = capital
L = labor

→ given a certain technology – represented by $F(.)$ – an amount of capital K and an amount of labor L are combined to produce some quantity of output X

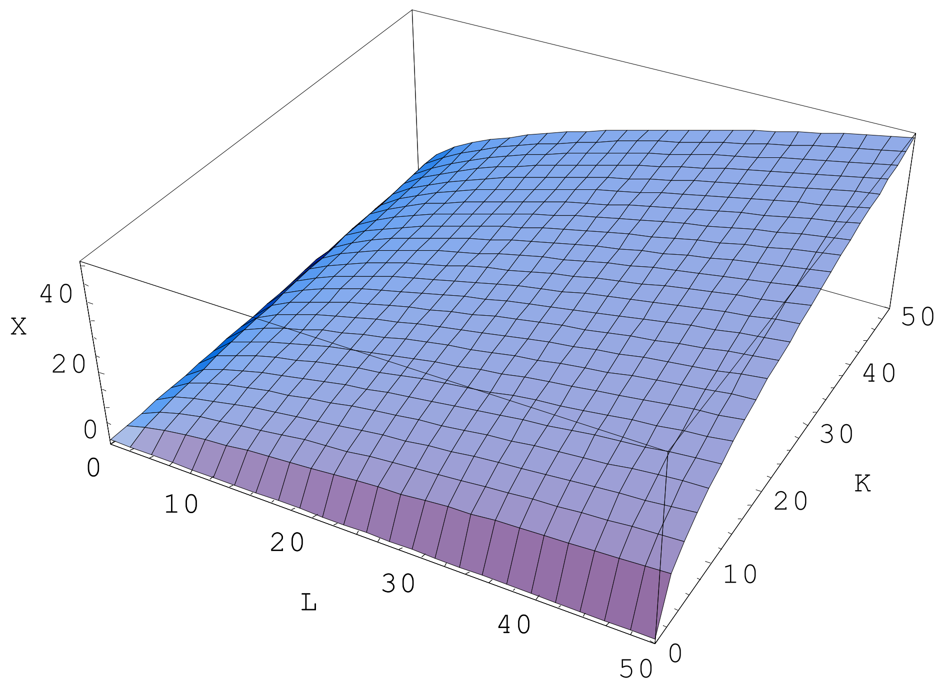
Assumption: production function is twice continuously differentiable

with $\frac{dX}{dL} > 0, \ \frac{dX}{dK} > 0$ and $\frac{d^2X}{dL^2} < 0, \ \frac{d^2X}{dK^2} < 0$

Specific example: Cobb-Douglas production function

$X = L^{\alpha_L} K^{\alpha_K}$ with $\alpha_L + \alpha_K = 1$

→ three-dimensional surface defined by this production function:

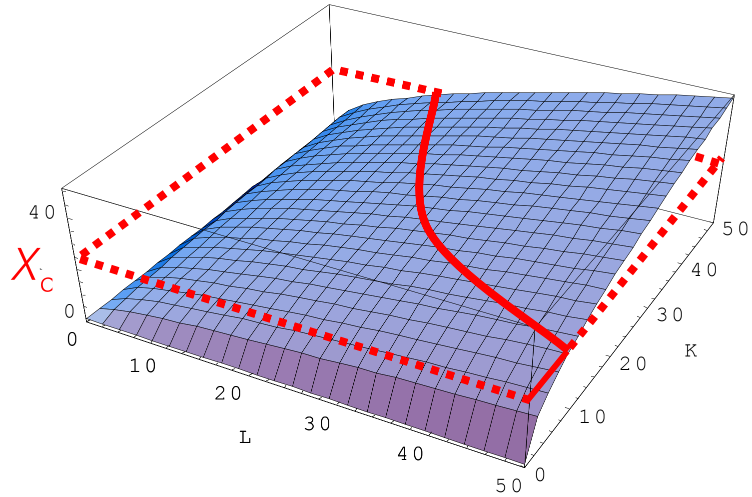


(here: $\alpha_L = .6$, $\alpha_K = .4$)

Convenient to consider two-dimensional slices of this 3-dimensional surface

1. (*L*, *K*)-plane:

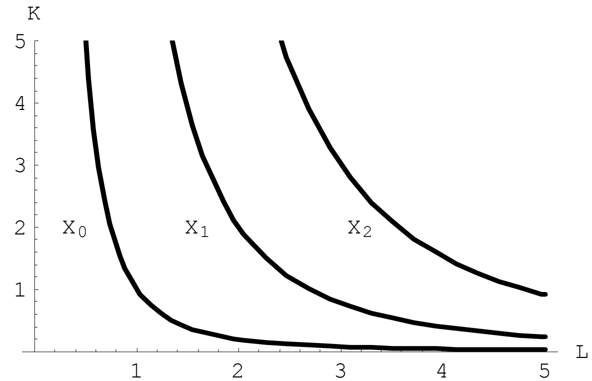
- keep output fixed and vary combination of inputs with which a constant output can be produced
- example: keeping X fixed at X_o



"isoquant": locus of all combinations of K and L which produce the same amount of X (here X_o).

- each production level can be represented by one isoquant:

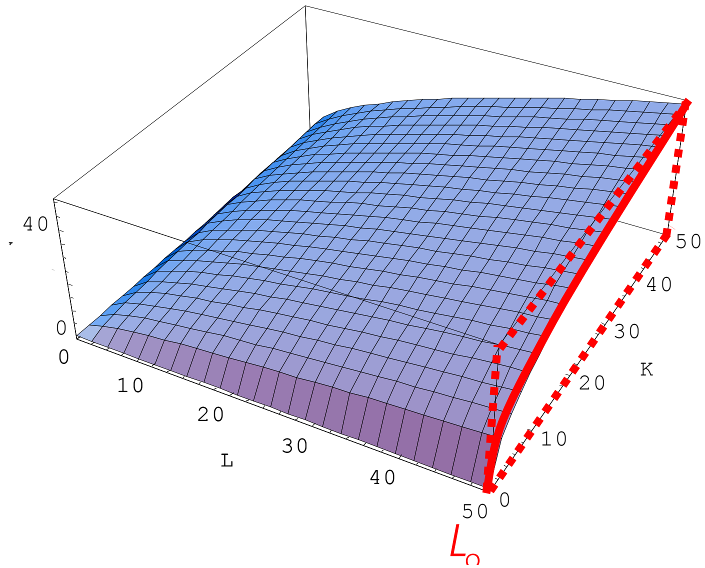
(where $X_o < X_1 < X_2$)



2. (*K*, *X*)- or (*L*, *X*)- plane

- keep one input factor fixed while the other input factor can be varied
- example: keeping L constant at L_o

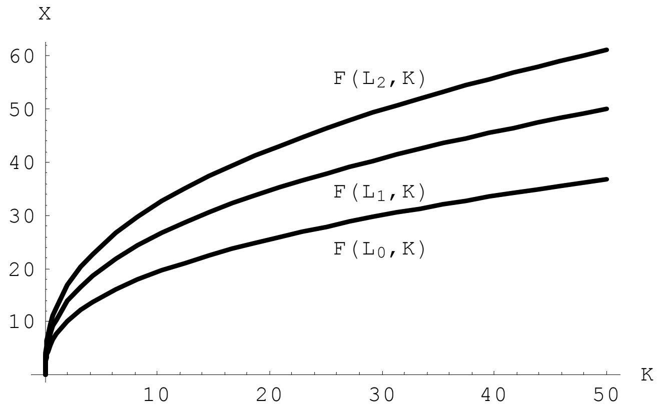
→ production function: $X = L_o^{\alpha_L} K^{\alpha_K}$



"total product curves":

output X as a function of one input (here K),

keeping the other input constant (here L at L_o)



- one total product curve for each level of L
- for any level of K, output is higher, the larger is the level of L (where $L_o < L_1 < L_2$)
- equivalently: total product curves when K is kept constant

- total product curves have two important characteristics:

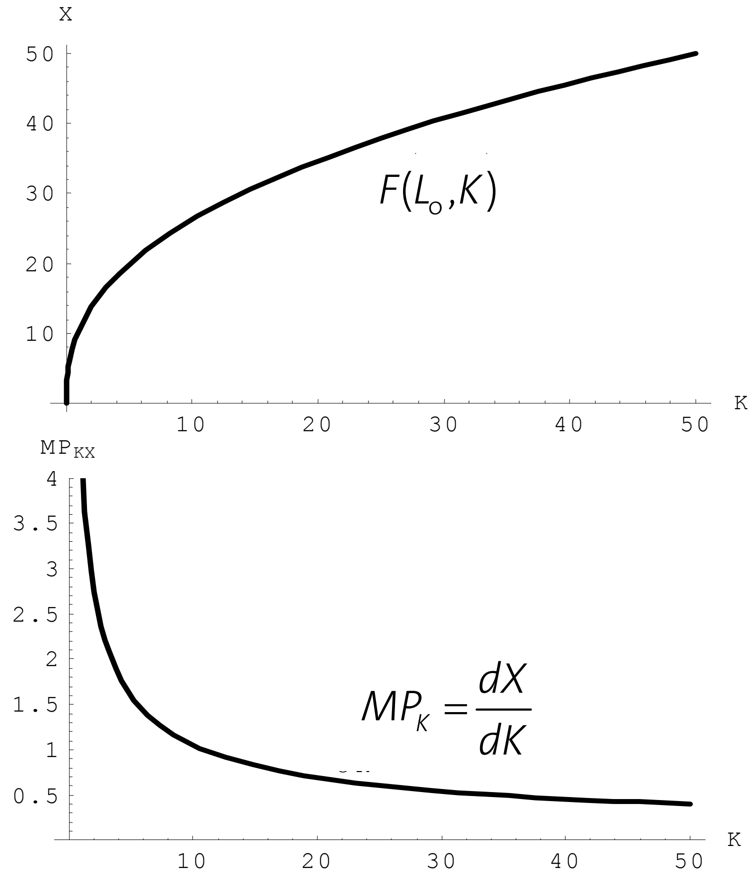
  1. start at the origin → K is **essential for production** (the same applies to L): $F(L,0)=F(0,K)=0$
  2. **diminishing returns**

  (slope of total product curve
  = marginal product of labor in production:

  $$MP_K = \frac{dX}{dK} = \alpha_K L^{\alpha_L} K^{\alpha_K - 1}):$$

  → given the input of L is constant, employing additional units of K increases X, but the rate of increase of X falls with rising levels of K → MP_K falls

  → production function exhibits "diminishing returns" in capital
  (equivalently for labor)



2.2. Returns to scale

Question: how does output change if all input factors are varied by the same proportional amount?

Definition (returns to scale):

Let $\lambda > 0$. The function $X=F(K, L)$ is said to be **homogeneous of degree *k*** if $\boxed{\lambda^k F(K,L) = F(\lambda K, \lambda L)}$.

- if ***k*=1**: prod. function is said to be homogeneous of degree 1 (linearly homogeneous) and production is characterized by CRS
- if ***k*<1:** decreasing returns to scale (DRS); if ***k*>1:** increasing returns to scale (IRS)

Often assumed: CRS $\rightarrow$ if $F(K, L)$ exhibits CRS, then doubling both inputs leads to doubling of output

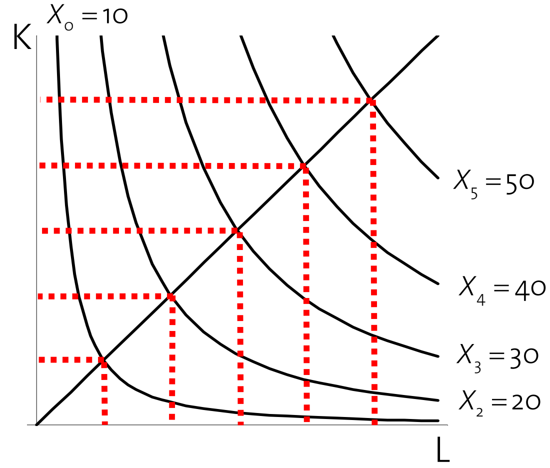
$\rightarrow$ Cobb-Douglas production function $X = L^{\alpha_L} K^{\alpha_K}$ (homogeneous of degree $k = \alpha_L + \alpha_K = 1$):

$$(\lambda L)^{\alpha_L} (\lambda K)^{\alpha_K} = \lambda^{\alpha_L} L^{\alpha_L} \lambda^{\alpha_K} K^{\alpha_K} = \lambda^{\alpha_L + \alpha_K} L^{\alpha_L} K^{\alpha_K} = \lambda X$$

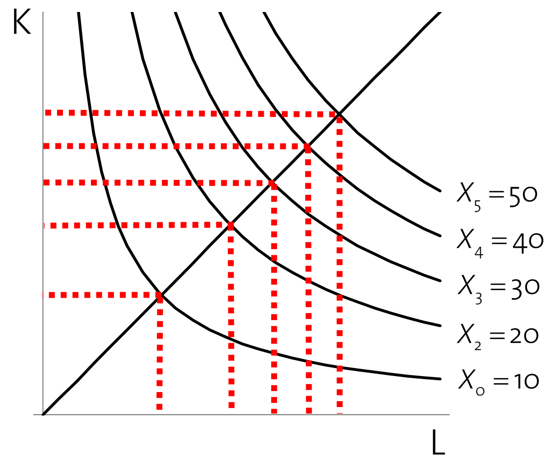
Implications of Homogeneity:

1. **spacing between isoquants**: the degree of homogeneity determines the spacing of the isoquants

- CRS: spacing between isoquants remains the same as doubling both inputs leads to a doubling of output: →

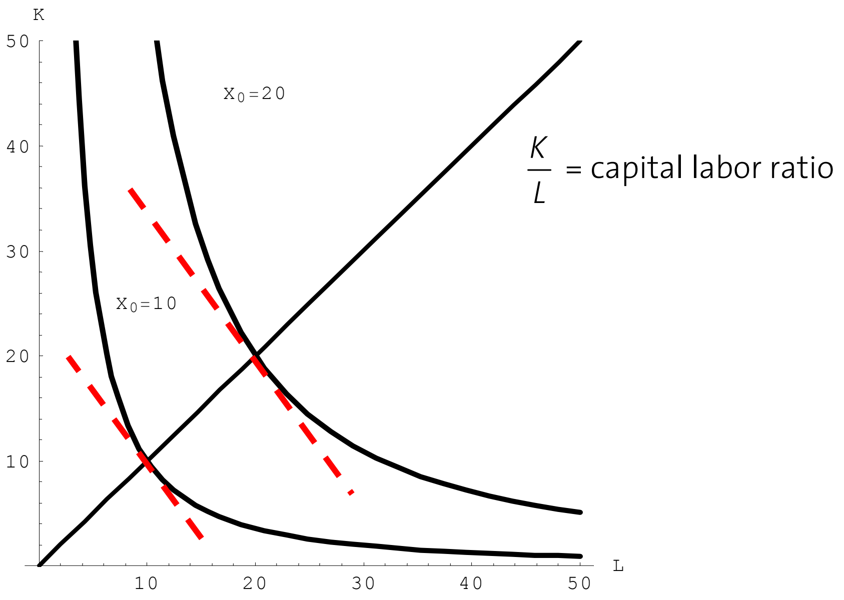


- IRS: spacing between isoquants falls as doubling of output is achieved with less than doubling of both inputs →



- DRS: ...

2. **slope of isoquants constant along a ray through the origin**



Notice:

- from **1.** and **2.** follows for CRS: once one isoquant is known, all other isoquants can be derived
- no conflict between diminishing returns (in one factor) and CRS.

  (diminishing returns: focus on isolated variations of one input factor, CRS: total factor variations)

2.3. Equilibrium for a single producer

typical producer: maximizes profits in a competitive environment

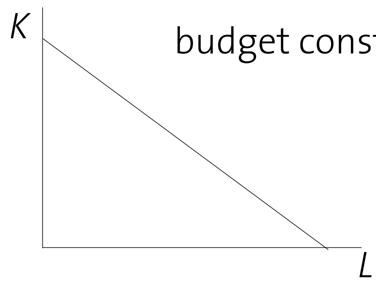
→ maximizing output subject to cost constraint:

$$\max_{K,L} \; X = F(K,L) \qquad \text{st.} \qquad C_o \geq wL + rK \qquad \text{with} \quad w = \text{wage rate}, \; r = \text{rental price of capital}$$

(*w* and *r*: constant from the perspective of the individual producer)

$C_o \geq wL + rK$ = budget constraint → total costs $w\,L+r\,K$ must not exceed a fixed amount C_o

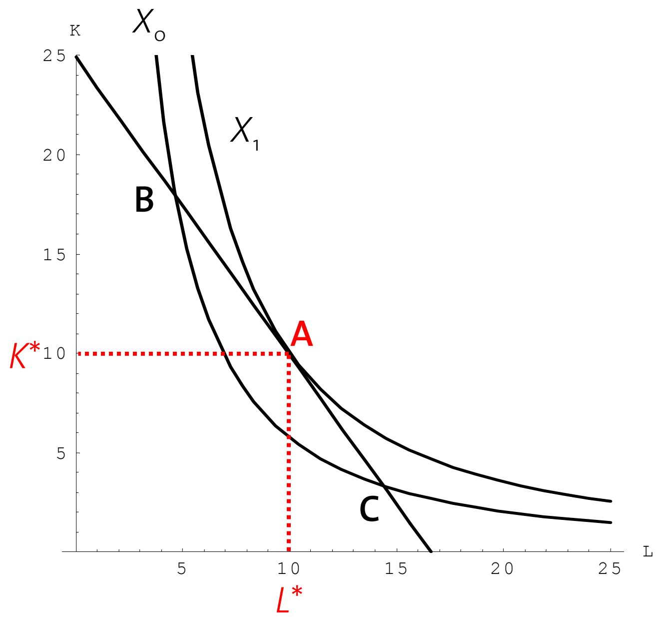
intuition: producer will employ as many inputs as possible to maximze output → $C_o = wL + rK$



budget constraint (slope: $-\frac{w}{r}$)

Equilibrium:

- outputs X_o and X_1, (i.e. points A, B and C,) can both be realized with the budget C_o
- But: only **A** can be optimal since both B and C imply a lower output ($X_o < X_1$)



→ optimal production plan (L^*, K^*): slope of budget constraint = slope of isoquant

(efficiency condition)

- slope of budget constraint: $-\frac{w}{r}$
- slope of isoquant:

totally differentiate the production function to get: $dX = \frac{\partial X}{\partial K} dK + \frac{\partial X}{\partial L} dL$

along an isoquant (dX=0): → $\left.\frac{dK}{dL}\right|_{dX=0} = -\frac{\frac{\partial X}{\partial L}}{\frac{\partial X}{\partial K}}$ = slope of isoquant

= "marginal rate of substitution"

intuition: If L is reduced by one unit (dL = -1), the amount of capital must be increased by $dK = \frac{\partial X / \partial L}{\partial X / \partial K}$ to keep output X constant.

- In equilibrium : $\frac{w}{r} = \frac{\frac{\partial X}{\partial L}}{\frac{\partial X}{\partial K}}$ (slope of budget constraint = slope of isoquant)

Remember: for homogeneous production functions all isoquants have the same slope along any constant capital-labor ratio $\frac{K}{L}$

Important implication:

→for any given wage-rental ratio, the optimal capital-labor ratio, $\frac{K^*}{L^*}$, will be constant regardless of the level of output

→the optimal capital-labor ratio is thus a function of the wage-rental ratio only (does not depend on the level of output)

Algebraic derivation of efficiency condition:

Lagrangian function (assuming constraint holds with equality)

$$\mathrm{L} = F(K,L) + \mu(C_o - wL - rK)$$

- necessary first-order conditions:

(1) $\frac{\partial \mathrm{L}}{\partial K} = \frac{\partial X}{\partial K} - \mu r = 0$ (2) $\frac{\partial \mathrm{L}}{\partial L} = \frac{\partial X}{\partial L} - \mu w = 0$ $\frac{\partial \mathrm{L}}{\partial \mu} = C_o - wL - rK = 0$

- from equation (1) and (2) : $\frac{w}{r} = \frac{\frac{\partial X}{\partial L}}{\frac{\partial X}{\partial K}}$

2.4. The two-good, two-factor model

Consider the following simple general-equilibrium model that will be used in the subsequent chapters:

- two commodities: X and Y
- produced with two input factors: K and L
- using the following technologies: $X = F_X(K_X, L_X)$

  $Y = F_Y(K_Y, L_Y)$

  - both production functions are linearly homogeneous and increasing in both inputs
  - both inputs are essential, i.e. $F_X(0,L_X) = F_X(K_X,0) = F_Y(0,L_Y) = F_Y(K_Y,0) = 0$

- fixed supplies of capital and labor: $\overline{K} = K_X + K_Y$

  $\overline{L} = L_X + L_Y$ (implicit: full employment)

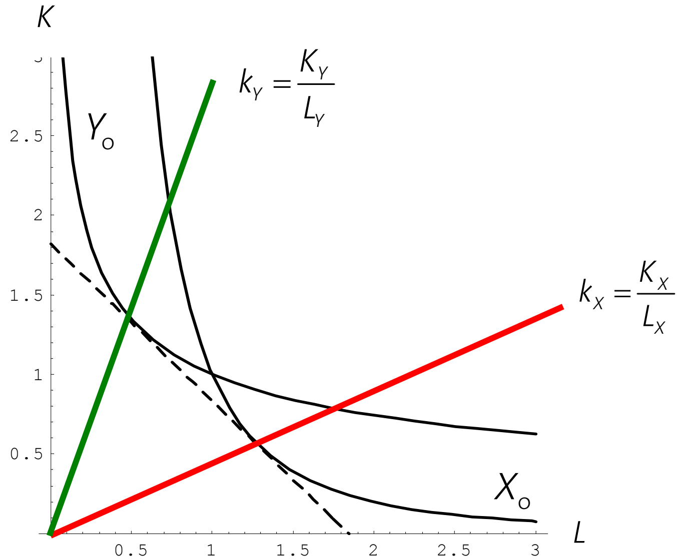
- X and Y differ in their factor intensities:

  for any given wage-rental ratio, <u>X is assumed to be labor intensive and Y to be capital intensive</u> (strong factor intensity hypothesis)

Definition (factor intensity): Let $k = \frac{K}{L}$ and assume factor prices are fixed.

If $k_Y > k_X$ at those factor prices, Y is said to be capital intensive and X is said to be labor intensive.

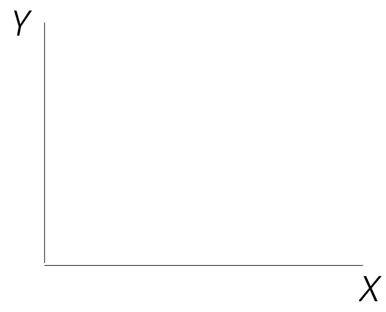
Graphically:



2.5 Shape of the production possibility frontier

Production possibility frontier (PPF; also called transformation curve):

locus of all efficient production points in the (*X*, *Y*)-plane

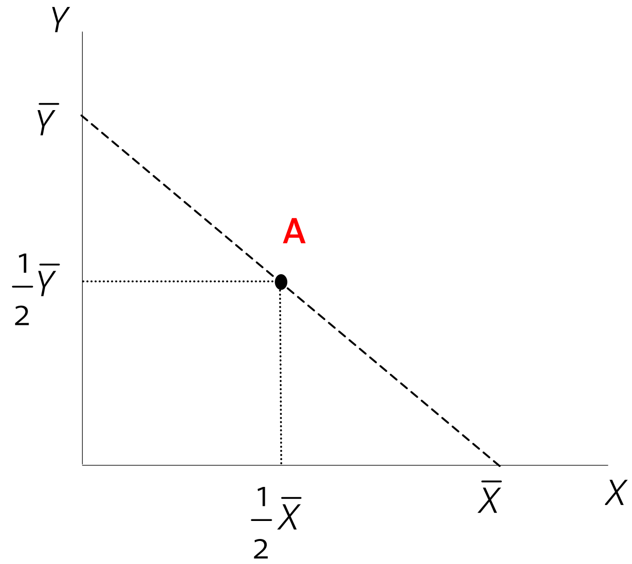


efficient production points:

- engineering efficiency: all producers maximize output for any inputs given
- market efficiency: taking the production of one good as given, factors are allocated such that the maximal possible amount of the other good is produced

Construction of PPF:

- all necessary information contained in
    - production functions, resource constraints
    - efficiency condition
- end points of PPP (only one good is produced):
    - $(0,\bar{Y})$ with $\bar{Y} = F_Y(\bar{K},\bar{L})$
    - $(\bar{X},0)$ with $\bar{X} = F_X(\bar{K},\bar{L})$



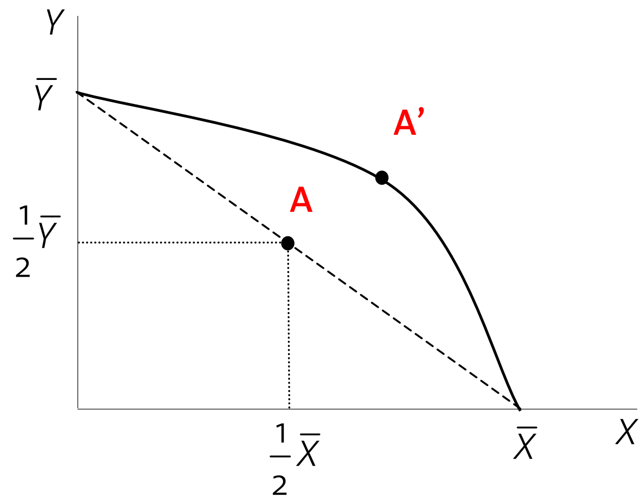
- if both goods are produced:
    - assume: half of each factor allocated to each product → **A** → efficient?

- as it is assumed that factor intensities differ between X and Y

  → production can be increased by reallocating some labor to production of X and some capital to Y

    → **A'**

  → PPF is concave to the origin (despite CRS in both sectors)

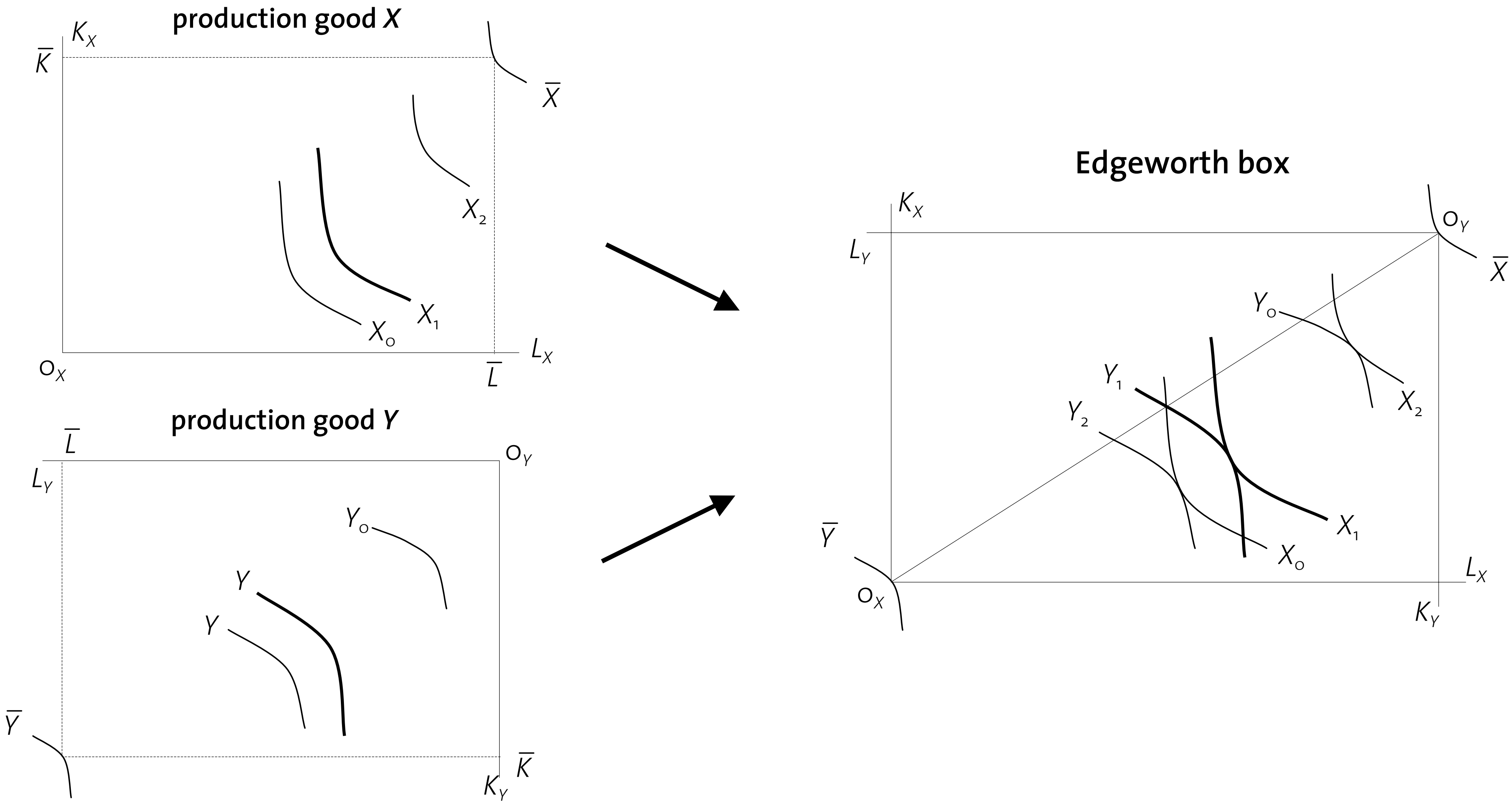


- if production technologies were identical: straight line

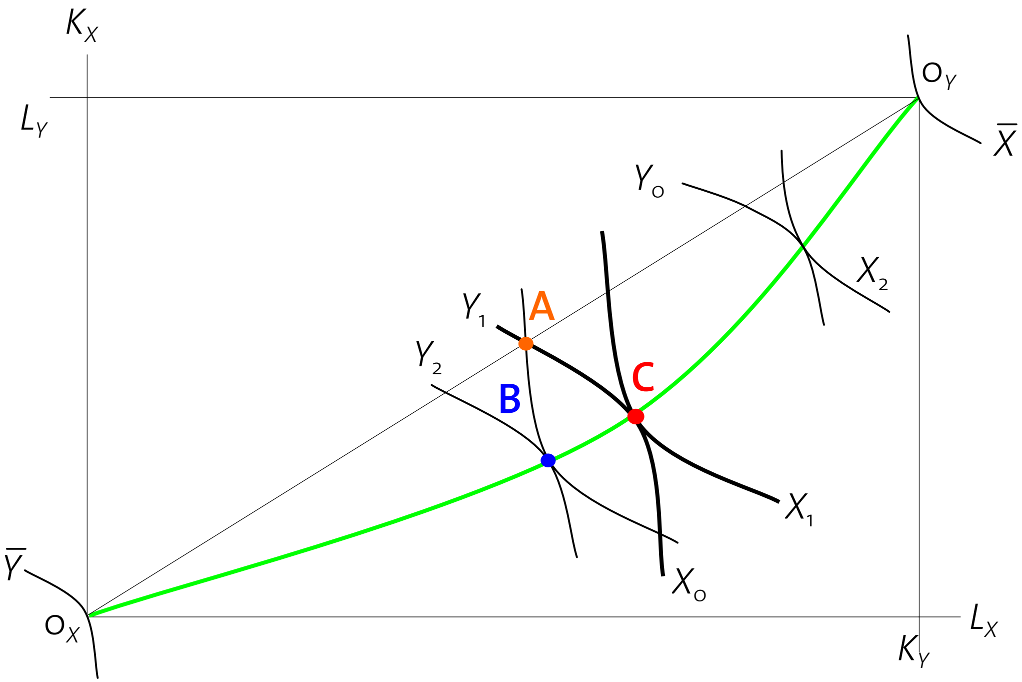
- terminology:

  the set of all feasible production points (bordered by the axes and the PPF) is a convex set.

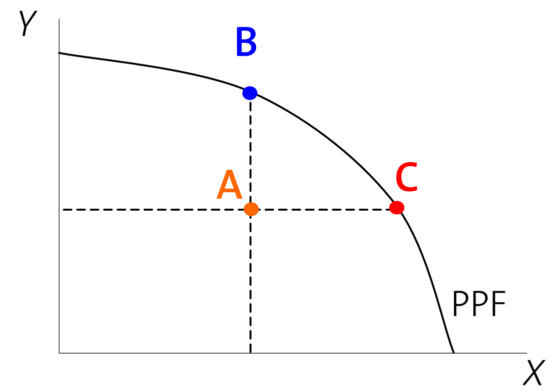
More rigorous graphical derivation:



Edgeworth Box and PPF:



→



A : inefficient production plan

B, C : efficient production plans

efficiency locus: all allocations of factors that represent market efficiency

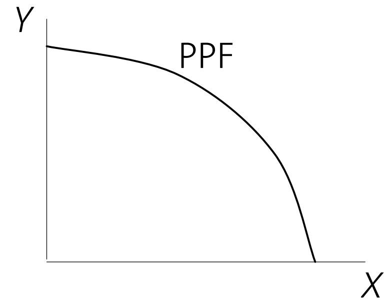
- allocations for which – given the production of one good – the amount produced of the other good is maximal
- along the efficiency locus: $MRS_X = MRS_Y$

Analytical illustration: PPF for a simplified set-up with only one input factor L

→ The production functions are $Y = L_Y^{\alpha_Y}$ and $X = L_X^{\alpha_X}$, $0 < \alpha_Y, \alpha_X < 1$

→ The resource constraint is $\overline{L} = L_X + L_Y$.

→ from substituting we get: $Y = \left(\overline{L} - X^{\frac{1}{\alpha_X}} \right)^{\alpha_Y}$



2.6. Competitive equilibrium

Consider a competitive market economy. Two questions arise which will be answered below:

(1) Does production actually take place on the PPF (i.e. is production efficient)?

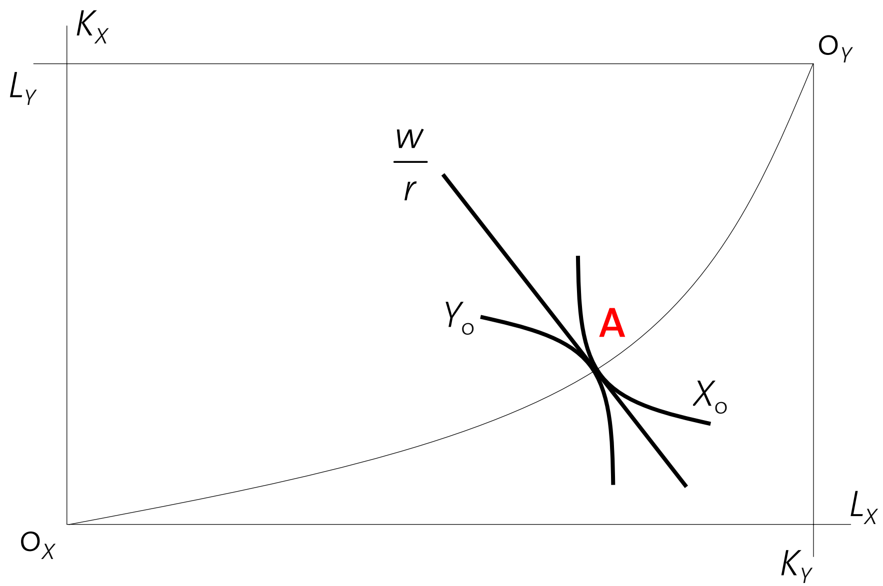
(2) If so, what point on the PPF is realized?

Answer to question (1): Does production actually take place on the PPF (i.e. is production efficient)?

- yes, if industries are **competitive** and face the **same factor prices**
- Reasoning:
  - for an efficient production structure (efficient allocation of inputs across the two sectors), the slope of isoquants (*MRS*) must be identical in both sectors: $MRS_X = MRS_Y$
  - output maximization subject to a cost constraint requires that the slope of the isoquant (*MRS*) equals the factor price ratio: $MRS_Y = w/r$ and $MRS_X = w/r$.

$\rightarrow MRS_X = MRS_Y = \frac{w}{r}$

$\rightarrow$ efficient production plan: **A**



Answer to question (2): If so, what point on the PPF is realized?

- a point on the PPF where the commodity price ratio $p = p_X / p_Y$ is tangent to the PPF, if **factor and commodity markets are competitive** and industries face the **same factor prices**.
- Reasoning: Competitive markets and profit maximization imply that the value marginal product of each factor (marginal product · output price) equals its respective price:

sector X: $p_x \dfrac{dX}{dL_X} = w$ (5) sector Y: $p_Y \dfrac{dY}{dL_Y} = w$ (7)

$p_x \dfrac{dX}{dK_X} = r$ (6) $p_Y \dfrac{dY}{dK_Y} = r$ (8)

Profit Maximization:

Firms maximize profits (revenues minus costs) given output and factor prices (e.g. sector X):

$\max\limits_{L_X, K_X} \quad \Pi_X = p_X X(L_X, K_X) - wL_X - rK_X$ where Π_X = profits in sector X

First order necessary conditions: $\dfrac{d\Pi}{dL_X} = 0 \Leftrightarrow p_X \dfrac{dX}{dL_X} = w$; $\dfrac{d\Pi}{dK_X} = 0 \Leftrightarrow p_X \dfrac{dX}{dK_X} = r$

Dividing equation (5) by (7), i.e. (6) by (8) gives:

$$\frac{p_X}{p_Y} = \frac{\frac{dY}{dL_Y}}{\frac{dX}{dL_X}} = \frac{\frac{dY}{dK_Y}}{\frac{dX}{dK_X}}$$

Because factors are in fixed supply, $dL_Y = -dL_X$ and $dK_Y = -dK_X$ we may write:

$$\frac{p_X}{p_Y} = \frac{\frac{dY}{-dL_X}}{\frac{dX}{dL_X}} = \frac{\frac{dY}{-dK_X}}{\frac{dX}{dK_X}} = -\frac{dY}{dX} =: MRT$$

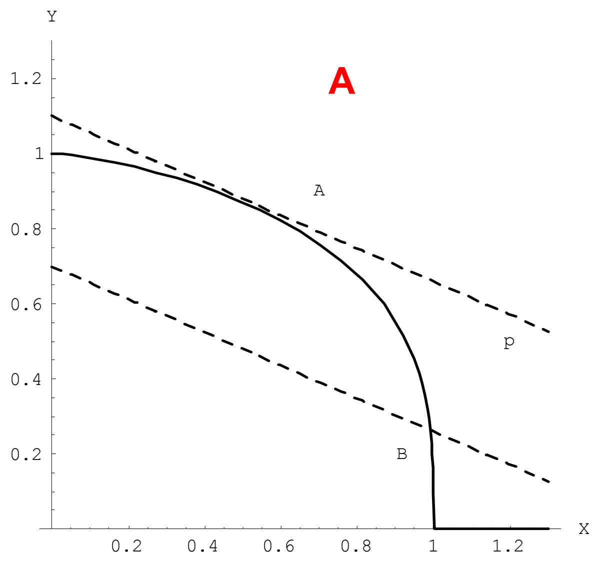
where the marginal rate of transformation (*MRT*) is the slope of the PPF.

This result holds for competitive and undistorted economies!

Graphically: Production occurs where the **price ratio is tangent to the PPF ($p = MRS$):**

- If world prices are given by $p = {}^{p_X}\!/\!_{p_Y}$: economy selects point **A**.
- In **A** the value of national output given world prices ($= p_X X + p_Y Y$) is maximized.

  This does not hold true for point B, for example.
- point **A** corresponds to a specific point within the Edgeworth box (on the contract curve), which implies a unique factor price ratio ${}^{w}\!/\!_{r}$.



→ in a trading economy, commodity prices on world markets determine commodity supplies,

→ commodity supplies in turn determine factor demands and hence factor prices.

Notice : economic causality is as follows: $p \rightarrow$ commodity supply → factor demand → factor prices

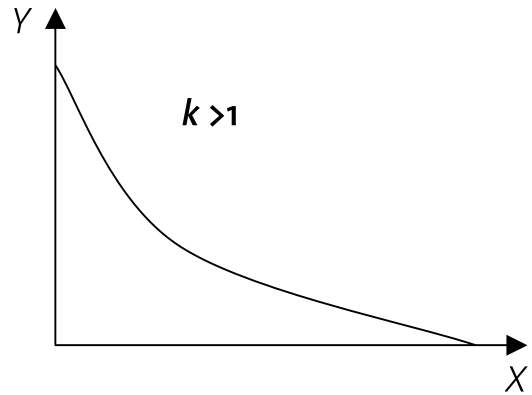
2.7. Increasing returns to scale

- typical industries with increasing returns to scale (IRS): aircraft and mainframe computer
- In 2.5: was shown that for CRS differences in factor intensities make the PPF concave (convex set of feasible production points)
- Here: will be shown that IRS make the PPF convex (set of feasible production points is non-convex)
- for simplicity: consider an economy with one input factor only
- production functions: $X = L_X^k$ with $k > 1$ → IRS (homogeneous of degree k>1)
  $Y = L_Y$ → CRS
- $\overline{L} = L_X + L_Y$

PPP for IRS in *X*:

- from substitution:

$$X = (L - L_Y)^k \quad \Leftrightarrow \quad L_Y = L - X^{\frac{1}{k}} \quad \Leftrightarrow \quad Y = L - X^{\frac{1}{k}}$$



(convexity of PPF would be reinforced if both sectors exhibited IRS)

Two important implications of IRS:

(1) Even two completely identical economies can realize gains from trade.

(2) IRS is incompatible with perfect competition and hence another theory must be used to explain trade under IRS (alternatively externalities may be used).